Constraints on the Quark Correlation Matrix from Equations of Motion

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We show that the Politzer theorem on the equations of motion implies strong constraints on the quark correlation matrix, restricting the number of independent distribution functions that characterize the internal structure of the nucleon. Then we draw some important consequences from this result. First of all, we suggest an alternative method for determining transversity. Secondly, we predict the Q^2-dependence of some azimuthal asymmetries. Lastly, we illustrate the origin of the main contributions to some twist-3 distribution functions.

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The theoretical expressions for inclusive reaction cross sections at high momentum transfers are typically of the form[1]

$$d\sigma \propto \mathcal{S}_1 \otimes \mathcal{S}_2 \otimes .... \otimes \mathcal{S}_m \otimes \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes .... \otimes \mathcal{H}_n, \qquad (1)$$

where the symbol "$\otimes$" means convolution and/or matrix product. The $\mathcal{S}_k$ (k = 1, ... m) denote "soft" functions, describing either the internal structure of the initial hadrons participating in the reaction, or the parton fragmentation into the final detected hadrons. On the other hand, the $\mathcal{H}_l$ (l = 1, ... n) describe the "hard" scattering, where the active partons are involved. Eq. (1) is known as the factorization property. As regards semi-inclusive reactions - which are receiving an increasing attention from spin physicists - this property was proven many years ago for back-to-back hadron pair production in $e^+ - e^-$ collisions[2] and recently also for semi-inclusive deep inelastic scattering (SIDIS) and Drell-Yan[3], which involve tranverse momentum dependent (tmd) "soft" functions. Moreover one sometimes adopts the handbag[4, 5] approximation, neglecting[6], either the interactions between such partons and the spectator ones after the hard scattering process[7] (in reactions like deep inelastic), or between the active quark and the spectator partons of other hadrons involved in the reactions (*e. g.*, in Drell-Yan). In this approximation the "soft" factors may be encoded in the so-called correlation and/or fragmentation matrix[4].

Mulders and Tangerman[4] (MT) used the Politzer theorem[8], concerning the equations of motion, to establish constraints between quark-quark and quark-quark-gluon correlation matrices. Here we draw from that theorem more dramatic consequences on the tmd quark-quark correlation matrix, showing that, in the framework of QCD, it fulfils an inhomogeneous Dirac equation. This strongly restricts the number of independent elements in that matrix, while uniquely fixing a given energy scale that appears in the parametrization of some of the "soft" functions. In turn this suggests, among other things, an alternative, promising method for extracting the transversity function from data. Moreover, the result allows to predict the Q^2-dependence for azimuthal asymmetries[7-11]. Lastly, a new light is cast on some twist-3 functions, generalizing a result about $g_2(x)$[12].

We denote the correlation matrix by Φ, which we define as

$$\Phi_{ij}(p; P, S) = \int \frac{d^4x}{(2\pi)^4} e^{ipx} \langle P, S|\bar{\psi}_j(0)\mathcal{L}(x)\psi_i(x)|P, S\rangle. \qquad (2)$$

Here $|P, S\rangle$ is a nucleon state with a given four-momentum P and Pauli-Lubanski (PL) four-vector S. Moreover

$$\mathcal{L}(x) = \mathrm{P}exp\left(-ig\int_0^x \lambda_a A^a_\mu(z)dz^\mu\right) \qquad (3)$$

is the gauge link operator[4], g, λ_a and A^a_μ being respectively the strong coupling constant, the Gell-Mann matrices and the gluon fields. The link operator depends on the choice of the integration path, say $\mathcal{P}$, which has to be fixed so as to make a physical sense[5, 11, 13]. According to previous treatments[4, 13], we define the path - to be denoted by $\mathcal{P}_+$ - as a set of three pieces of straight line, from the origin to $x_{1\infty} \equiv (+\infty, 0, \mathbf{0}_\perp)$, from $x_{1\infty}$ to $x_{2\infty} \equiv (+\infty, x^+, \mathbf{x}_\perp)$ and from $x_{2\infty}$ to $x \equiv (x^-, x^+, \mathbf{x}_\perp)$. Here we have adopted a frame - to be used throughout this letter - whose z-axis is taken along the nucleon momentum; moreover we have introduced light cone coordinates.

Two remarks are in order. Firstly, we have omitted the flavor and color indices of the quark, according to the MT notation; similarly, we shall drop the flavor index from the "soft" functions. Secondly, for T-odd functions[14], the path $\mathcal{P}_+$ is suitable, for example, for SIDIS, while for Drell-Yan one has to consider the one through $x^- = -\infty$[11, 15, 16, 17] to be denoted by $\mathcal{P}_-$.

Now we deduce an equation for the correlation matrix in the framework of QCD. To this end, first of all, we recall the Politzer theorem[8], *i. e.*,

$$\langle h|\mathcal{F}(\psi)(i\not{D} - m_q)\psi(x)|h\rangle = 0. \tag{4}$$

Here m_q is the quark rest mass, $|h\rangle$ a hadronic state, $\mathcal{F}(\psi)$ a functional of the quark field and D_μ the covariant derivative, $D_\mu = \partial_\mu - ig\lambda_a A^a_\mu$. The result (4) survives renormalization. Furthermore, taking into account the path $\mathcal{P}_+$ in $\mathcal{L}(x)$, the covariant derivative implies

$$\begin{aligned}\mathcal{L}(x)(i\not{D} - m_q)\psi(x) &= (i\not{\partial} - m_q)\left[\mathcal{L}(x)\psi(x)\right] \\ &+ g\lambda_a \not{B}^a_\perp(x)\mathcal{L}(x)\psi(x).\end{aligned} \tag{5}$$

Here

$$B^{a\mu}_\perp(x) = \int_{x^-}^\infty dy^- H^{a\mu}_-(y^-, x^+, \mathbf{x}_\perp) \tag{6}$$

and $H^{a\mu}_- = \partial_- A^{a\mu} - \partial^\mu A^a_-$. From eqs. (4) and (5) - setting $\mathcal{F}(\psi) = \bar{\psi}(0)\mathcal{L}(x)$ and $|h\rangle = |P, S\rangle$ - we get

$$\langle P, S|\bar{\psi}_j(0)(i\not{\partial} - m_q)\left[\mathcal{L}(x)\psi_i(x)\right]|P, S\rangle = g\beta_{ij}, \tag{7}$$

where

$$\begin{aligned}\beta_{ij} &= \beta_{ij}(x; P, S) \\ &= -\langle P, S|\bar{\psi}_j(0)\lambda_a \not{B}^a_\perp(x)\mathcal{L}(x)\psi_i(x)|P, S\rangle.\end{aligned} \tag{8}$$

Moreover

$$\int d^4x e^{ipx}\not{\partial}\left[\mathcal{L}(x)\psi(x)\right] = -i\not{p}\int d^4x e^{ipx}\mathcal{L}(x)\psi(x), \tag{9}$$

since $\psi(x)$ must vanish at infinite distances, in order to guarantee four-momentum conservation. Fourier transforming both sides of eq. (7), and applying the result (9), we get

$$(\not{p} - m_q)\Phi(p; P, S) = g\tilde{\beta}(p; P, S), \tag{10}$$

where

$$\tilde{\beta}(p; P, S) = \int \frac{d^4x}{(2\pi)^4} e^{ipx}\beta(x; P, S). \tag{11}$$

and the β matrix is defined through eq. (8). The solution to eq. (10) is of the type

$$\Phi(p; P, S) = (\not{p} + m_q)\Psi, \tag{12}$$

where

$$\begin{aligned}\Psi = \delta(p^2 - m_q^2)\Psi_f(p; P, S) + \\ g\tilde{\beta}(p; P, S)(p^2 - m_q^2 - i\epsilon)^{-1}.\end{aligned} \tag{13}$$

Here the first term of eq. (13) corresponds to the correlation matrix of an on-shell quark in the QCD improved parton model. On the contrary, the second term takes into account quark offshellness and interaction dependence. Hermiticity, Lorentz invariance and parity invariance for Φ bind Ψ to be a linear combination of 5 Dirac operators, *i. e.*,

$$\Psi = \sum_{i=1}^5 \Gamma_i B_i(p^2, P\cdot p), \tag{14}$$

where B_i are Lorentz invariant real functions, while

$$\begin{aligned}\Gamma_1 &= 1, \quad \Gamma_2 = \gamma_5 \not{S}_{q\parallel}, \quad \Gamma_3 = \gamma_5 \not{S}_{q\perp}, \\ \Gamma_4 &= -i\not{P}_\perp/\mu, \quad \Gamma_5 = -i\not{P}_\perp \gamma_5 \not{S}_q/\mu.\end{aligned}$$

Here S_q is the PL vector of the active quark,

$$\begin{aligned}S_{q\parallel} &= (S_q\cdot n_+)n_- + (S_q\cdot n_-)n_+, \\ S_{q\perp} &= S_q - S_{q\parallel}\end{aligned}$$

and $n_\pm$ are lightlike vectors, such that $n_+\cdot n_- = 1$ and whose space components are directed along the momentum of the quark. Moreover

$$P_\perp = P - (P\cdot n_+)n_- - (P\cdot n_-)n_+.$$

Lastly μ is a parameter with the dimensions of a momentum, introduced for dimensional reasons and to be determined below.

Now we insert eq. (14) into (12), observing that for $i = 1$ to 3 the operators $\not{p}\Gamma_i$ are twist 2 and therefore interaction independent, while for $i = 4$ and 5 such operators are T-odd and therefore depend on interactions, since they describe interference effects[18]. Therefore, comparing eq. (13) with eq. (14) yields

$$\Psi_f = \sum_{i=1}^3 \Gamma_i B'_i, \qquad g\tilde{\beta} = \sum_{i=4}^5 \Gamma_i B'_i, \tag{15}$$

where

$$\begin{aligned}B_i &= B'_i\delta(p^2 - m_q^2), \qquad i = 1, 2, 3, \\ &= B'_i(p^2 - m_q^2 - i\epsilon)^{-1}, \quad i = 4, 5.\end{aligned}$$

Notice that $\tilde{\beta}$ depends crucially on the path $\mathcal{P}$, see eqs. (5) and (6); in particular it changes sign according as to whether $\mathcal{P} = \mathcal{P}_+$ or $\mathcal{P}_-$, as expected for T-odd functions[11].

Now we introduce the projections, integrated over p^-, of the correlation matrix over the Dirac components[4], *i. e.*,

$$\Phi^\Gamma(p^+, \mathbf{p}_\perp; P, S) = \frac{1}{2}\int dp^- tr(\Gamma\Phi), \tag{16}$$

where Γ is a Dirac operator and $p \equiv (p^-, p^+, \mathbf{p}_\perp)$. Inserting eqs. (12) and (13) into (16), and closing, according

to eq. (11), the integration path in the upper complex half-plane, we get

$$\Phi^\Gamma(p^+, \mathbf{p}_\perp; P, S) = \frac{1}{4p^+} tr\left[\Gamma(\not{p}_0 + m_q)(\Psi_{f0} + 2\pi i g\tilde{\beta}_0)\right]. \quad (17)$$

Here Ψ_{f0} and $\tilde{\beta}_0$ are, respectively, the functions Ψ_f and $\tilde{\beta}$ calculated at $p = p_0$, where $p_0 \equiv [(m_q^2 + \mathbf{p}_\perp^2)/2p^+, p^+, \mathbf{p}_\perp]$. On the other hand, the projections over the Dirac components may be expressed according to the usual notations of the tmd functions[4]. For example, we have

$$\begin{aligned} \Phi^{\gamma^+} &= f_1, & (18)\\ \Phi^{\gamma^5\gamma^+} &= \Lambda g_{1L} + \lambda_\perp g_{1T}, & (19)\\ \Phi^{\gamma^5\gamma^l\gamma^+} &= S^l h_{1T} + \pi_\perp^l\left(\Lambda h_{1L}^\perp + \lambda_\perp h_{1T}^\perp\right), & (20)\\ \Phi^{\gamma^l\gamma^+} &= \pi_\perp^l h_1^\perp, & (21)\\ \Phi^{\gamma^i n_{\perp i}} &= \pi_\perp^i n_i f_{1T}^\perp. & (22) \end{aligned}$$

Here $\Lambda = MS^+/P^+$ is the nucleon light cone helicity, M the nucleon mass, $\lambda_\perp = -p_\perp \cdot S/\mu$ and $p_\perp = p - (P \cdot p/M^2)P$, in our frame $p_\perp \equiv (0, 0, \mathbf{p}_\perp)$. Moreover $\pi_\perp = p_\perp/\mu$ and $n_\perp$ is such that $n_\perp^2$ = -1, $n_\perp \cdot P = n_\perp \cdot p_\perp = 0$. Comparing such relations with the projections (17) along the same Dirac components, we obtain some important results. First of all, we identify

$$\begin{aligned} f_1 &= B'_{1,0}, \quad g_{1L} = B'_{2,0},\\ h_{1T} &= B'_{3,0}, \quad h_1^\perp = 2\pi B'_{4,0},\\ f_{1T}^\perp &= 2\pi B'_{5,0}, \end{aligned}$$

where the $B'_{i,0}$ are the functions B'_i calculated at $p = p_0$. It follows that the 5 functions listed above - f_1, g_{1L}, h_{1T}, $h_1^\perp$ and $f_{1T}^\perp$ - are Lorentz invariant, and therefore they depend on the Bjorken variable x, on Q^2 and on $p_\perp^2$. Incidentally, if we consider the correlation matrix (12) in the limit of $g \to 0$ and integrate both sides of that equation over p^-, we get (up to a normalization factor) the spin density matrix of a free, on-shell quark, i. e.,

$$\begin{aligned} \rho &= 1/2(\not{p}_0 + m_q)\Psi_{f0}, & (23)\\ \Psi_{f0} &= f_1 + g_{1L}\gamma_5\not{S}_{q\parallel} + h_{1T}\gamma_5\not{S}_{q\perp}. & (24) \end{aligned}$$

Secondly we establish some relations among the traditional[4] "soft" functions, as expected, since our parametrization consists of 5 independent functions, instead of the 12 found by MT. For example, from eqs. (19) and (20), it follows

$$g_{1T} = h_{1T}^\perp \frac{1-\epsilon_1}{1-\epsilon_2} = \frac{|\mathbf{p}|}{\mu} h_{1T}(1-\epsilon_1). \quad (25)$$

Here $\mathbf{p}$ is the momentum of the quark. Moreover

$$\begin{aligned} 1-\epsilon_1 &= cos\theta - \alpha(1 - cos\theta), & (26)\\ 1-\epsilon_2 &= A - \alpha(A+1), & (27) \end{aligned}$$

$A = \sqrt{2}xP^+/|\mathbf{p}| - cos\theta$, $sin\theta = |\mathbf{p}_\perp|/|\mathbf{p}|$ and $\alpha = [(m_q^2 + \mathbf{p}^2)^{1/2} - |\mathbf{p}|]/m_q$. In deriving result (25), we have taken into account that S_q does not coincide with S: indeed, one has $S \equiv (0, \mathbf{S})$ in the *nucleon* rest frame and $S_q \equiv (0, \pm\mathbf{S})$ in the *quark* rest frame[19, 20], with $\mathbf{S}^2 = 1$.

Furthermore, g_{1T} and $h_{1T}^\perp$ can be interpreted as probability densities, as well as h_{1T}, provided they are appropriately normalized. Therefore we have to fix

$$\mu = |\mathbf{p}|. \quad (28)$$

This result was also found starting from a simple model[21] for T-odd functions, based on the interference between two different partial waves[7, 22, 23].

It is to be noticed that, by considering the projections (17) over all the Dirac components, one obtains several relations of the type (25) among the usual T-even tmd functions[4]. Several such relations can be established by considering the projections over the Dirac components of eq. (23), which immediately shows that the independent T-even tmd functions amount to f_1, g_{1L} and h_{1T}. All other T-even functions are related to these through frame dependent multiplicative factors, as can be seen, for example, from eqs. (25).

Each of the 5 independent functions characterizing the correlation matrix is related to an u-channel helicity amplitude for quark-nucleon elastic scattering[24, 25]:

$$\begin{aligned} &f_1 = \phi_1^u + \phi_2^u, \qquad g_{1L} = \phi_1^u - \phi_2^u,\\ &h_{1T} = \phi_3^u, \qquad sin\theta h_1^\perp = \phi_4^u,\\ &sin\theta cos\varphi f_1^\perp = \phi_5^u. \end{aligned}$$

Here $cos\varphi = -p_\perp \cdot S/|\mathbf{p}_\perp|$ and the amplitudes ϕ_l^u (l = 1 to 5) are obtained by analytical continuation of the s-channel amplitudes

$$\begin{aligned} \phi_1^s &= -i\langle + + |T - T^\dagger| + +\rangle,\\ \phi_2^s &= -i\langle + - |T - T^\dagger| + -\rangle,\\ \phi_3^s &= -i\langle + - |T - T^\dagger| - +\rangle,\\ \phi_4^s &= -i\langle + + |T - T^\dagger| + -\rangle,\\ \phi_5^s &= -i\langle + + |T - T^\dagger| - +\rangle. \end{aligned}$$

We have denoted by T the T-matrix and by $|\Lambda, \lambda\rangle$ the helicity states, where Λ and λ are respectively the nucleon and quark helicity. Notice that pure symmetry considerations, independent of the details of the interaction, would yield 6 independent helicity amplitudes, since in this case only parity and time reversal invariance can be applied[24]; therefore we conclude that a further constraint is implied by gauge invariance, as we have assumed.

To conclude, we sketch some consequences of our theoretical results.

A) Eq. (20) implies the following expression of the transversity:

$$h_1(x) = \int d^2p_\perp \left[h_{1T}(x, \mathbf{p}_\perp^2) + \lambda_\perp^2 h_{1T}^\perp(x, \mathbf{p}_\perp^2)\right]. \quad (29)$$

We point out that this expression contains a term - the second one of eq. (29) - which is frame dependent, since $\lambda_\perp = sin\theta cos\varphi$ and $h_{1T}^\perp$ contains the factor $(1-\epsilon_2)$, see eq. (25). To illustrate this term, consider a transversely polarized nucleon. The quark longitudinal polarization, due in this case to the transverse momentum, is magnified by the boost from the quark rest frame. This additional, frame dependent, polarization has a component along the transverse momentum, see the second term of eq. (20). Furthermore eq. (29), together with eqs. (25) to (28), suggests an alternative method for determining the transversity of a nucleon. Indeed, g_{1T} can be conveniently extracted from double spin asymmetry[20, 26, 27] in SIDIS with a transversely polarized target. This asymmetry is expressed as a convolution of the unknown function with the usual, well-known fragmentation function of the pion. Therefore the method appears more convenient than the usually proposed one[9], based on the Collins effect[28] in single spin SIDIS asymmetry, since in this case one is faced with a convolutive product of two unknown functions, h_{1T} and the Collins function.

B) Eq. (28) determines the Q^2-dependence of the azimuthal asymmetries whose theoretical expressions in the formalism of the correlation matrix contain the factor μ^{-1}. Typical cases are the two above mentioned SIDIS asymmetries, for which we predict a decrease like Q^{-1}, contrary to the current literature[4]. Our result follows immediately from eq. (28), by calculating the quark momentum in the Breit frame where the the time component of the virtual photon momentum vanishes. Analogously, we predict a decrease like Q^{-2} for the polarized SIDIS and Drell-Yan azimuthal asymmetries, which, in the formalism of the correlation matrix, result in convolutive products of two T-odd functions[29]. In particular, in the case of Drell-Yan, data[10] exhibit such an azimuthal asymmetry, which, at least for $\mathbf{p}_\perp^2 << Q^2$, can be conveniently interpreted in the framework of that formalism[30]; but the Q^2 dependence is consistent with our prediction[31] and not with the MT statement, $\mu = M$.

C) Eqs. (12) and (13) imply that, in the handbag approximation, the T-even twist 3 "soft" functions[4] are proportional either to $|\mathbf{p}_\perp|$ or to m_q. As a consequence of this fact, the most significant contributions to the $\mathbf{p}_\perp$-integrated functions $g_T = g_1 + g_2$, h_L and e_1 come from non-handbag diagrams, generalizing the result of Efremov and Teryaev for g_2[12, 32].

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